On the Number of Partitions of Sets and Natural Numbers

Abstract

In this article, we first state some relations about the number of partitions of a set under some particular conditions and then we give a new relation about the number of partitions of an n-set, i.e., Bell number B(n). Finally, we give some formulas to count partitions of a natural number n, i.e., partition function P(n).

1 Introduction

Partitions of sets and natural numbers have been a very attractive subject during the recent decades. Partitions play important roles in such diverse areas of mathematics as combinatorics, Lie theory, representation theory, mathematical physics, and the theory of special functions. Because of some applications of this subject, mathematicians have given some formulas in this regard. Up to now, the number of partitioning of sets and natural numbers was considered by some authors; for more details about partitions of sets and other related subjects, see for example [8], [10], [15], [17], [18], [19], [20], [21], and [22], and for partitions of natural numbers and other related concepts, see for example……

In this article, using some elementary tools of combinatorial analysis- see [6], [7], [11], [16], and [23]- we give some alternative formulas for theses problems, considering also some special cases.

Về số phân hoạch của các tập hợp và các số tự nhiên

Tóm tắt

Trong bài viết này, trước hết chúng ta phát biểu một số hệ thức về số phân hoạch của một tập hợp trong một số điều kiện cụ thể và sau đó chúng ta đưa ra một hệ thức mới về số phân hoạch của tập n, tức là, số Bell, B (n). Cuối cùng, chúng ta đưa ra một số công thức để đếm số phân hoạch của một số tự nhiên n, tức là, hàm phân hoạch P (n).

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## 2 Partitions of sets

Definition 2.1. A partition of a set A is any sequence of subsets A\,...,Am of A, such that (J "L1 Ai = A and Ai fi Aj = 0. Vi = j.

The number of partitions of a set of size n ( n-set) is called the Bell number, in honor of famous mathematician “Eric Temple Bell” (1883-1960), and denoted by B(n). By convention we agree that B(0)=1. Using Definition 2.1, for n=1,2,3, we have B(1) = 1, B(2) = 2, B(3)=5 and so on.

In the following lemma, we state a relation for the number of partitions of an n-set such that in every partition we have at least a subset with n - j elements, j = 1, 2,..., [|], n > 2.

[redacted]

Lemma 2.1 . Let B(n\n — j) be the number of partitions of a set with n elements in which there exists at least a subset with n — j elements, j = 1,2,..., [|], Then for n > 2, we have

………………………

where

……………………………..

Proof. Let j e {1,2, ...,[|] — 1}. Because n — j > |, the statement “at least a subset with n — j elements” is equivalent to “exactly a subset with n — j elements”. n — j elements from n elements can be selected in,,,, ways. But the remaining j elements can be partitioned in B(j) ways. Hence…..

But

\n] _ / if n is ever,

U 2 \ if n is odd

Therefore, if n is odd, we have n — [|] >

[redacted]

|, hence

…………………………..

If n is even, we cannot partition the remaining | elements unconditionally, because when the set is partitioned in two subsets with | elements, \ ^ ^ ^ partitions are counted twice and therefore in this case, we have

……………………………..

Example 2.1 . Find the number of partitions of a set with 5 elements in which there exists exactly a subset with

(a) 4 elements;

(b) 3 elements.

Example 2.2 . Find the number of partitions of a 4-set with at least a subset with 2 elements( B(4|2)).

Solution. Using Lemma2.1, we have ……………….

These 9 partitions are as follows:

……………………..

Now, we state two other lemmas about the number of partitions of a set with n elements in which all subsets have m elements. We indicate this number by Pm(n).

Lemma 2.2 . If mln(i.e. there exist a natural number k such that n=km), then

………………………..

Proof. The number of ways that we can distribute n elements to — subsets,

………………….

such that in every subset we have m elements, is

…………………………..

Hence, by using definition of multinomial coefficients, this lemma is proved. □

Lemma 2.3 . Let P'm(n) indicate the

[redacted]

[redacted]

number of partitions of an n-set to maximal number of subsets with m elements. We have

……………………………

Proof. m[^\ is the greatest multiple of m that is less than or equal to n. These m[^\ elements are selected in ^ ^ ways. But the remaining (n — to[^]) elements must be partitioned arbitrarily. Thus, the relation is proved. □

Lemma 2.4 . Let m\n and Pm\ni,...n-(n) indicate the number of partitions of an n-set into subsets of size m, such that in j subsets from m special elements of the set, there exist in numbers n\, 77.2, •••, % elements , j = 1, 2,..., Yll=i ni = m. We have

…………………………

Proof. The number of ways in which m elements can be distributed to j subsets, with numbers n1,n2, is

…………………….

But the number of elements of these j subsets isn't still m. Therefore n — m of the remaining elements must be partitioned to j + 1 subsets, in numbers m —n1,...,m—nj, and finally n—jm. This number is ni=1 ( n km + ^1=0 ni y y m — nk J n0 = 0 or ( n m |. Now the n—jm elements ym — n1,m — n2,...,m — nj,n — jm J are remained that must be partitioned. Using Lemma 2.2 and the product axiom in combinatorial analysis, the relation is obtained.□

Theorem 2.1 . If m\n, then

…………………………………..

Proof. Consider m special elements of the set {1. 2. ...,n}, for instance These m elements can be distributed in ^

[redacted]

[redacted]

[redacted]

arbitrary subsets. Consider a case that m elements in j subsets are distributed in numbers n1,n2. ...,nj, j = 1, 2,YH=i ni = m• But the number of these cases is i'>m|ri1,ri2,...,rij(^). Now by using Lemma 2.4 and summing over j, j = 1,2,...,—, the proof is completed. □

Example 2.3 . Find the number of partitions of set {1. 2..... 9} in subsets vnth 3 elements,

(a) without any condition;

(b) 3 elements 1, 2, 3 are in 3 different subsets.

Solution. (a) Using Lemma 2.2, we have ……………………..

(b) The number is P3\1t1,1(9). Now, Using Lemma 2.4, we have

…………………………………

Theorem 2.2 . If n > 1, then

………………………………

Proof. By above lemmas, the proof is obvious. □

3 Partition function

Definition 3.1 . A partition of a natural number n is any non-increasing sequence of natural numbers whose sum is n.

In this section, we state some lemmas and theorems about P(n), the number of partitions of natural number n. By convention, we agree that P(0) = 1. It can be shown that P(1) = 1, P(2) = 2, P(3) = 3, P(4) = 5 and so on.

Lemma 3.1 . Let P(n|1, 2,...,m) be the number of partitions of a natural number n, such that each summand is at most m. Then…………

Proof. It is obvious that if m > n, then

[redacted]

condition “at most m” has not any restriction. Therefore in this case, we have P (n\1. 2.....m) = P (n). Now, let m < n. In this case the greatest summand is m. Because we can arrange summands from left to right, non-increasingly, if in a partition we have summand m, the remaining number (n — m), must be partitioned; but not arbitrarily. This number must be partitioned such that the greatest summand of this partition is m. These numbers are P (n — m\1. 2.....m).

If we have summand m — 1, the number n — (m — 1) must be partitioned such that any summand is not greater than m— 1. This number is P(n—(m—1)\1. 2..... m— 1). Continuing this method until the first summand in left partition is 1, and summing on the number of all cases, the relation will be obtained. □

Theorem 3.1 . We have

…………………………………..

Proof. It is stated that P(0) = P(1) = 1 and P(2) = 2. Now, let n > 3. If the first summand is n — 1, then there is one case. Hence in general, if the first summand is n — i, i = 0. 1. ....n — 1, the remaining number i must be partitioned. Of course any summand can not be greater than n — i. But these numbers are P(i\ 1. 2. ....n — i). By summing on all cases, the proof is completed. □

Lemma 3.2 . Let P1(i; n) be the number of partitions of a natural number n such that in every partition we have i summands 1, exactly. Now………

[redacted]

[redacted]

Proof. P1(i ; n) has i summands 1, but the remaining number (n — i) must be partitioned such that we have not any summand 1. □

Lemma 3.3 . Let P1(0; n\2. 3.....m) indicate the number of partitions of number i without summand 1, such that summands can be 2. 3.....m. Now,

………………..

and also,

………………………………

Proof. It is obvious that P1(0; 0) = 1 and P1(0; 1) = 0. But

………………. Acting similar to Lemma 3.1, the proof is completed. □

Theorem 3.2 . We have

………………………..

Proof. In each partition of natural number n, for the number of summands

1, there exist n + 1 cases: 0.1.....n. By definition of P1(i; n), the proof is completed. □

Example 3.1 . Find the number of partitions of 9, by using

(a) Theorem 3.1;

(b) Theorem 3.2.

Solution. (a)Using Theorem 3.1, we have

………………………

Hence P(9) = 30. b) Using Theorem 3.2, we have

P(9) = Pi(0; 9) + Pi(1; 9) + ... + Pi(9; 9),

and by Lemma 3.2,

P (9) = Pi(0;9) + Pi(0;8) +... + Pi(0;0).

But by Lemma 3.3, we have

……………………

[redacted]

[redacted]

Hence P(9) = 30.
A
Result 3.1 .
(a) Let P(nlA),A c IN, denote the number of partitions of n such that all summands belong to set A and P(n|3s G A) denote this number providing to we have at least a summand s belonging to A . We have ……………
(b) The number of partitions such that the greatest summand is a multiple of
………. if ….. if …...
Proof. Using the definitions, the proof is obvious. □
Remark 3.1 .
We have…………..
Lemma 3.4 . Let P12 ... m(n1,n2,...,nm; n) be the number of partitions of n such that in every partition, summands j, j=1,2,...,m appear nj times. We have
…………………………………..
Proof. We discard summands 1, 2,...,m from partition of n and then we partition the remaining number, n — Y^mm=i j.nj, such that in these recent partitions the summands 1, 2, ... , m do not appear. □
Theorem 3.3 . We have
………………………………….
Proof. For calculating P(n), we sum on P1 2 ,... ,m(n1,n2, ...,nm; n) over all…..

Example 3.2 . Find the number of partitions of 8 such that,
(a) at least a summand is less than 3;
(b) the greatest summand is even;
(c) summands 1 and 2 appear 2 times and 1 time, respectively.

[redacted]

| | |
|---|---|
| Solution. (a) Using Result 3.1 part (a), we have<br>P(8\at least a summand is less than 3) = P(8) — P(813, 4,..., 8).<br>But by Lemma 3.1, we have…………..<br><br>(b) By Result 3.1 part (b),………..The greatest summand is even…………….<br>(c) The desired number is P1)2(2,1; 8). By Lemma 3.4, We have………….<br>This partition is 8 = 4 + 2 + 1 + 1. A | [redacted] |